RESOLUTION OF INDECOMPOSABLE INTEGRAL FLOWS ON A SIGNED GRAPH

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Abstract. It is well-known that each nonnegative integral flow of a directed graph can be decomposed into a sum of nonnegative graph circuit flows, which cannot be further decomposed into nonnegative integral sub-flows. This is equivalent to saying that indecomposable flows of graphs are those graph circuit flows. Turning from graphs to signed graphs, the indecomposable flows are much richer than that of ordinary unsigned graphs. The present paper is to give a complete description of indecomposable flows of signed graphs from the viewpoint of resolution of singularities by introducing covering graphs.

A real *flow* (also known as *circulation*) on a graph or a signed graph (a graph with signed edges) is a real-valued function on oriented edges such that the net inflow to each vertex is zero. An *integral flow* is a flow whose values are integers. There are many reasons to be interested in flows on graphs; an important one is their relationship to graph structure through the analysis of *(conformally) indecomposable flows*, that is, integral flows that cannot be decomposed as the sum of two integral flows having the same sign on each edge (both ≥ 0 or both ≤ 0). It is well known, and an important observation in the theory of integral network flows, that the indecomposable flows are identical to the *circuit flows*, which have value 1 on the edges of a graph circuit (=cycle) and 0 on all other edges. Extending the theory of indecomposable integral flows to signed graphs, which is studied in [7] by algorithmic method, led to a remarkable discovery that there are, besides the anticipated circuit flows (which are already more complicated in signed graphs than in ordinary graphs), many "strange" indecomposable flows with elaborate structure not describable by signed-graph circuits.

In this article we characterize this structure again by the method of sign-labeled covering graphs, lifting each vertex and each edge of a signed graph to two vertices and two edges respectively of a sign-labeled covering graph, and lifting each indecomposable flow to a simple cycle flow in the sign-labeled covering graph. We think of the lifting as a combinatorial analog of resolution of singularities in algebraic geometry. The strange indecomposable flows are singular phenomena, which we resolve by lifting them (blowing up overlapped edges) to ordinary cycle flows in a covering graph. Comparing to the algorithmic approach in [7], the present paper hints a connection (at least conceptually) between graph theory and covering spaces of algebraic topology and resolution of algebraic geometry. We believe that this connection is useful to study gain graphs [14] that are more complicated than signed graphs.

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It is characterized that indecomposable integral flows of signed graphs are characteristic vectors of certain so-called Eulerian cycle-trees (see Definition 9, Theorem 13 and Theorem 15 below). The properties of Eulerian cycle-trees lead to the following half-integer scale conformal decomposition:

> *Every nontrivial integral flow of an oriented signed graph can be conformally decomposed into a positive half-integer (perhaps integer but not always integer) linear combination of signed-graph circuit flows.*

1. Graphs and signed graphs

Graphs. A graph is a system $G = (V, E)$, with vertex set V and edge set E, such that each edge $x \in E$ is associated with a multiset $\mathrm{End}(x)$ of two vertices, called the end-vertices of x; the edge x is called a *link* if the two vertices of $\mathrm{End}(x)$ are distinct, and is called a *loop* if the two vertices are identical. Let x be an edge and $\mathrm{End}(x) = \{u, v\}$; we say that x is *incident* with u and v, or u and v are *adjacent* by x; we write (u, x) and (v, x). A tricky technical point is that this notation does not distinguish the two end-vertices of a loop; we take an easy way out by treating $\{(u, x), (v, x)\}$ as a multiset when $u = v$.

A *walk of length n* in a graph $G = (V, E)$ is a pair $W = (\boldsymbol{u}, \boldsymbol{x})$ of functions

$$\boldsymbol{u} : \{0, 1, \ldots, n\} \to V, \quad \boldsymbol{x} : \{1, 2, \ldots, n\} \to E$$

such that the end-vertices of the edge $\boldsymbol{x}(i)$ are the vertices $\boldsymbol{u}(i-1)$ and $\boldsymbol{u}(i)$. We write $u_i = \boldsymbol{u}(i)$, $x_i = \boldsymbol{x}(i)$, and

$$W = u_0x_1u_1x_2 \ldots u_{n-1}x_nu_n.$$

A walk is said to be *closed* if $n \geq 1$ and $v_0 = v_n$ and *open* otherwise. A subsequence of the form $u_ix_{i+1}u_{i+1}x_{i+2} \ldots u_{j-1}x_ju_j$ is called a *sub-walk* of W. A walk is called a *trail* if there are no repeating edges, except the initial and terminal vertices for closed walk (such a trail is called a *closed trail*). A walk is called a *path* (or *simple walk*) if there are no repeating vertices (and subsequently there are no repeating edges), except the initial and terminal vertices for closed walk (such paths are called *closed paths*). As usual, we call a closed path a *cycle*. A *graph circuit* is a cycle; the name comes from the fact that the cycles of a graph form the circuits of the graphic matroid whose elements are the edges of the graph.

An *isthmus* (or *cut-edge*) is an edge whose deletion increases the number of connected components. A *cut-vertex* is a vertex whose deletion, with all incident edges, increases the number of components, or that supports a loop and is incident with at least one other edge. A *block* is a maximal subgraph without cut-vertices. Thus, a loop or isthmus or isolated vertex is a (trivial) block. We call blocks *adjacent* if they have a common vertex (which is necessarily a cut-vertex). An *end-block* is a block adjacent to exactly one other block. For the notions of graph theory that we didn't define, we refer to the books [2, 3, 10]

Signed graphs. A *signed graph* $\Sigma = (V, E, \sigma)$ consist of an ordinary graph $G = (V, E)$ and a function $\sigma : E \to \{-1, 1\}$ (signs are multiplied rather than added), called the *sign function* of Σ. The *sign* of a walk $W = u_0x_1u_1 \ldots x_nu_n$ in Σ is the product

$$\sigma(W) = \prod_{i=1}^{n} \sigma(x_i).$$

In particular, a cycle has a sign, which is either positive or negative. A subgraph or its edge set is said to be *balanced* if every cycle in the subgraph (induced by the edge set) is positive.

A *signed-graph circuit* in a signed graph is a subgraph (or its edge set) of one of the following three types:

(i) A positive cycle, said to be of *Type I*.
(ii) A pair of negative cycles whose intersection is a single vertex, said to be of *Type II* (also known as a *contrabalanced tight handcuff*).
(iii) A pair of vertex-disjoint negative cycles together with a simple path of positive length (called the *circuit path*) that connects the two cycles and is internally disjoint from the cycles, said to be of *Type III* (also known as a *contrabalanced loose handcuff*).

The signed-graph circuits form the circuits of a matroid on the edge set of the signed graph [12]. An ordinary unsigned graph can be considered as a signed graph whose all edges are positive. When all edges are positive, every cycle is positive and the only signed-graph circuits are those of Type I, that is, the graph circuits.

Orientation. A *bi-direction* of a graph (a concept introduced by Edmonds [8]) is a function from the vertex-edge pairs to the sign group. One thinks of an edge at its one end-vertex with either a positive sign $+1$ as having an arrow directed away from the end-vertex, or a negative sign -1 as having an arrow directed towards the end-vertex. Thus each edge is assigned two arrows, one at each of its end-vertices. Technically, a bi-direction may be described by a multi-valued function $\varepsilon : V \times E \to \{-1, 0, +1\}$ such that (i) $\varepsilon(u, x) = 0$ if $u \notin \mathrm{End}(x)$, (ii) $\varepsilon(u, x)$ is a nonzero value if x is a link, (iii) if x is a loop then $\varepsilon(u, x)$ is a multiset of two (possibly identical) nonzero values.

An *orientation* of a signed graph Σ is a bi-direction ε on its underground graph such that for each edge x with end-vertices u, v (possibly x is a loop so that $u = v$),

$$\sigma(x) = -\varepsilon(u, x)\varepsilon(v, x), \quad x = uv.$$

Then a positive edge x has two arrows in the same direction along x thus indicating a direction along x as in an ordinary directed graph. A negative edge x has two opposite arrows that both point away from or both point towards the end-vertices. A signed graph together with an orientation is known as an *oriented signed graph*; see [5, 6, 13].

Let (Σ, ε) be an oriented signed graph throughout. A *source* in (Σ, ε) is a vertex u at which all edges are directed outwards, that is, $\varepsilon(u, x) = +1$ for all edges x at u. Conversely, if all edges at u point towards u, the vertex u is called a *sink*.

A walk $W = u_0x_1u_1x_2 \ldots u_{n-1}x_nu_n$ of length n on an oriented signed graph is said to be *coherent* at u_i if

$$\varepsilon(u_i, x_i) + \varepsilon(u_i, x_{i+1}) = 0,$$

that is, the order of W follows the orientations of edges at the vertex u_i. When W is a closed walk, we apply this definition to u_0 by taking subscripts modulo n, that is,

$$\varepsilon(u_n, x_n) + \varepsilon(u_0, x_1) = 0.$$

An open walk is said to be *directed* in (Σ, ε) if it is coherent at its every internal vertex. A closed walk is said to be *directed* in (Σ, ε) if it is coherent at every vertex, including at the initial and terminal vertex. A *direction* of W is a function ε_W with values either 1 or -1, defined for all vertex-edge pairs (u_{i-1}, x_i) and (u_i, x_i), such that

$$\varepsilon_W(u_{i-1}, x_i)\varepsilon_W(u_i, x_i) = -\sigma(x_i), \quad \varepsilon_W(u_i, x_i) + \varepsilon_W(u_i, x_{i+1}) = 0. \tag{1}$$

Every walk has exactly two opposite directions. A walk W with a direction ε_W is called a *directed walk*, denoted (W, ε_W).

Lemma 1. *The sign of a closed walk equals $(-1)^k$, where k is the number of times that the walk is incoherent at vertices, including the initial and terminal vertex.*

Proof. We perform a short calculation that applies to open as well as closed walks. Let $W = u_0x_1u_1 \ldots x_nu_n$ be a walk, either open or closed. Then

$$\begin{aligned}\sigma(W) &= \prod_{i=1}^{n} \sigma(x_i) = \prod_{i=1}^{n} \big(-\varepsilon(u_{i-1}, x_i)\varepsilon(u_i, x_i)\big) \\ &= -\varepsilon(u_0, x_1)\varepsilon(u_n, x_n) \prod_{j=1}^{n-1} \big(-\varepsilon(u_j, x_j)\varepsilon(u_j, x_{j+1})\big) \\ &= \begin{cases} (-1)^k\varepsilon(v_0, e_1)\varepsilon(v_n, e_n) & \text{if } W \text{ is open,} \\ (-1)^k & \text{if } W \text{ is closed,} \end{cases}\end{aligned}$$

where k is the number of times that W is incoherent at its internal vertices if W is an open walk. □

Let $S \subseteq E$ be an edge subset. A *reorientation* by S is an orientation ε_S obtained from ε by reversing the orientations of the edges in S and keeping the orientations of edges outside S unchanged. Thus ε_S is given by

$$\varepsilon_S(v, e) = \begin{cases} -\varepsilon(v, e) & \text{if } e \in S, \\ \ \ \varepsilon(v, e) & \text{if } e \notin S. \end{cases}$$

Sign-labeled covering graph. Let the end-vertices of each link edge of the graph $G = (V, E)$ be ordered arbitrarily and fixed. The *sign-labeled covering graph* of a signed graph $\Sigma = (V, E, \sigma)$ is an ordinary graph $\tilde{\Sigma} = (\tilde{V}, \tilde{E})$ with the vertex and edge sets

$$\tilde{V} = V \times \{\pm\}, \quad \tilde{E} = E \times \{\pm\},$$

defined as follows: If two vertices $u, v \in V$ are adjacent by an edge $x \in E$, then the vertices (u, α), $(v, \alpha\, \sigma(x))$ in $\tilde{V}$ are adjacent by an edge $\tilde{x}$ in $\tilde{E}$, and the vertices $(u, -\alpha)$, $(v, -\alpha\, \sigma(x))$ are adjacent by another edge $\tilde{x}^*$ in $\tilde{E}$. We denote $\tilde{x}$ by (x, β) (with the index β randomly selected) and $\tilde{x}^*$ by $(x, -\beta)$. For simplicity, we write

$$u^\alpha = (u, \alpha), \quad x^\beta = (x, \beta), \quad x^\beta = u^\alpha v^{\alpha\, \sigma(x)}, \quad x^{-\beta} = u^{-\alpha} v^{-\alpha\, \sigma(x)}.$$

There is no canonical way to choose the index β to label the edge between u^α and $v^{\alpha\, \sigma(x)}$ with x^β, when the edge x is a link between its two end-vertices u, v. However, if we choose one of two orders between the end-vertices u and v, say, $u \succ v$, we can choose labels as follows:

$$x^\alpha = u^\alpha v^{\alpha\, \sigma(x)}, \quad \alpha = +, -.$$

If so, we have $x^+ = u^+ v^{\sigma(x)}$ and $x^- = u^- v^{-\sigma(x)}$. Notice that the symbol x^β is just a sign-labeled edge in $\tilde{\Sigma}$, nothing to do with the sign $\sigma(x)$ of the edge x in Σ. All sign-labeled covering graphs are isomorphic, when various orders are selected for the end-vertices of links in the underlying signed graph.

There is a natural graph homomorphism $\pi : \tilde{\Sigma} \to \Sigma$, called the *projection* from $\tilde{\Sigma}$ to Σ, which is a pair of functions $\pi_V : \tilde{V} \to V$ and $\pi_E : \tilde{E} \to E$, defined by

$$\pi_V(u^\alpha) = u, \quad \pi_E(x^\beta) = x.$$

We usually write π_V and π_E simply as π. There is a canonical involutory but fixed-point free graph automorphism * of $\tilde{\Sigma}$, called the *augmentation* of $\tilde{\Sigma}$, defined by

$$(u^\alpha)^* = u^{-\alpha}, \quad (x^\beta)^* = x^{-\beta}.$$

When x is a negative loop at its unique end-vertex u, the edges x^+, x^- are two parallel edges in $\tilde{\Sigma}$ with the end-vertices u^+, u^-. We may think of $\tilde{V}$ as having a positive level $V^+ = \{u^+ \mid u \in V\}$ and a negative level $V^- = \{u^- \mid u \in V\}$. We shall see later that it is impossible to lift all edges of Σ to the same levels when Σ is unbalanced. A positive edge is lifted to two edges staying inside each level, but a negative edge is lifted to two edges crossing between the two levels.

The orientation ε on Σ can be lifted to an orientation $\tilde{\varepsilon}$ on $\tilde{\Sigma}$ as follows, called the *lift* of ε. Let $x \in E$ be an edge incident with a vertex $u \in V$, and let $\tilde{x}$ be a lift of x and be incident with a vertex $u^\alpha \in \tilde{V}$. Then we define

$$\tilde{\varepsilon}(u^\alpha, \tilde{x}) = \alpha\, \varepsilon(u, x). \tag{2}$$

Since $u^{-\alpha}$ is an end-vertex of the lifted edge $\tilde{x}^*$, then by definition we have

$$\tilde{\varepsilon}(u^{-\alpha}, \tilde{x}^*) = -\alpha\, \varepsilon(u, x).$$

Alternatively, if we have chosen an order between u and v for each link $x = uv$, say, $u \succ v$, then we define

$$\tilde{\varepsilon}(u^\alpha, x^\alpha) = \alpha\, \varepsilon(u, x), \quad \tilde{\varepsilon}(v^{\alpha\,\sigma(x)}, x^\alpha) = \alpha\, \sigma(x)\, \varepsilon(v, x). \tag{3}$$

It is easy to see that the two arrows on each lifted edge $\tilde{x}$ are the same, regardless of the sign of the edge x in Σ. In fact, for each edge x with end-vertices u, v (possibly $u = v$), we have

$$\tilde{\varepsilon}(u^\alpha, \tilde{x})\, \tilde{\varepsilon}(v^{\alpha\,\sigma(x)}, \tilde{x}) = \alpha\, \varepsilon(u, x)\, \alpha\, \sigma(x)\, \varepsilon(v, x) = -1.$$

Analogously, we have

$$\tilde{\varepsilon}(u^{-\alpha}, \tilde{x}^*)\, \tilde{\varepsilon}(v^{-\alpha\,\sigma(x)}, \tilde{x}^*) = (-\alpha\, \varepsilon(u, x))(-\alpha\, \sigma(x)\, \varepsilon(v, x)) = -1.$$

This means that the two arrows on each of the two lifted edges $\tilde{x}, \tilde{x}^*$ have the same directions; see Figure 1 for a link edge and Figure 2 for a loop edge. So $(\tilde{\Sigma}, \tilde{\varepsilon})$ is an ordinary directed unsigned graph. Note the covering graph $\tilde{\Sigma}$ is independent of the chosen order on the two

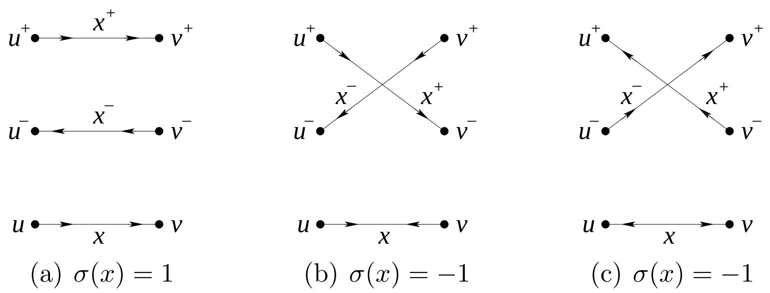


FIGURE 1. Lifting of a link edge and its orientation

end-vertices of each link edge in the sense that the covering graphs with respect to different orders are all graph isomorphic.

Two oriented edges $x, y \in E$ incident with a common vertex v, with the orientations $\varepsilon(v, x)$ and $\varepsilon(v, y)$, are said to be *coherent* at v if

$$\varepsilon(v, x)\, \varepsilon(v, y) = -1, \tag{4}$$

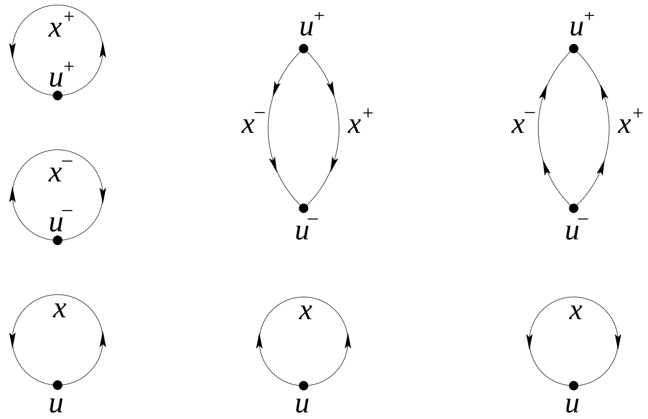


FIGURE 2. Lifting of a loop and its orientation

which is equivalent to $\varepsilon(v,x)+\varepsilon(v,y)=0$. We shall see that the lifting of orientations preserves coherence. If x and y are oriented edges with a common vertex v, with the orientations $\varepsilon(v,x)$ and $\varepsilon(v,y)$, then the concatenated lift of xvy to $\tilde{x}v^{\alpha}\tilde{y}$, with the orientations $\tilde{\varepsilon}(v^{\alpha},\tilde{x})$ and $\tilde{\varepsilon}(v^{\alpha},\tilde{y})$, is coherent at v. In fact,

$$\tilde{\varepsilon}(v^{\alpha},\tilde{x})\,\tilde{\varepsilon}(v^{\alpha},\tilde{y})=\alpha\,\varepsilon(v,x)\,\alpha\,\varepsilon(v,y)=-1. \tag{5}$$

Let ε_i be orientations on signed subgraphs Σ_i, $i=1,2$. The *coupling* of ε_1 and ε_2 is a function $[\varepsilon_1,\varepsilon_2]:E\rightarrow\{-1,0,1\}$, defined (for each edge x with its end-vertex u) by

$$[\varepsilon_1,\varepsilon_2](x)=\begin{cases}\;\;1 & \text{if } x\in\Sigma_1\cap\Sigma_2,\ \varepsilon_1(u,x)=\varepsilon_2(u,x),\\ -1 & \text{if } x\in\Sigma_1\cap\Sigma_2,\ \varepsilon_1(u,x)\neq\varepsilon_2(u,x),\\ \;\;0 & \text{otherwise.}\end{cases} \tag{6}$$

One may extend ε_i to Σ by requiring $\varepsilon(v,y)=0$ whenever the edge y in not incident with the vertex v in Σ_i. We always assume this extension automatically. Then alternatively,

$$[\varepsilon_1,\varepsilon_2](x)=\varepsilon_1(u,x)\,\varepsilon_2(u,x),\quad \text{where } u\in\mathrm{End}(x).$$

The lifted graphs $\tilde{\Sigma}_i$ are subgraphs of $\tilde{\Sigma}$ and $(\tilde{\Sigma}_i,\tilde{\varepsilon}_i)$ are oriented subgraphs of the oriented graph $(\tilde{\Sigma},\tilde{\varepsilon})$. Moreover, the lifting of orientations *preserves* the coupling, that is,

$$[\tilde{\varepsilon}_1,\tilde{\varepsilon}_2](x^{\beta})=[\varepsilon_1,\varepsilon_2](x) \tag{7}$$

for each lift x^{β} of an edge x. Indeed,

$$[\tilde{\varepsilon}_1,\tilde{\varepsilon}_2](x^{\beta})=\tilde{\varepsilon}_1(u^{\alpha},x^{\beta})\,\tilde{\varepsilon}_2(u^{\alpha},x^{\beta})=\alpha\,\varepsilon_1(u,x)\,\alpha\,\varepsilon_2(u,x)=[\varepsilon_1,\varepsilon_2](x).$$

Let $W=u_0x_1u_1x_2\ldots x_nu_n$ be a walk in Σ of length n with a direction ε_W. We may lift W to a walk $\tilde{W}$ (called a *lift* of W) in $\tilde{\Sigma}$ as follows: Select an initial vertex u^{α_0}; define

$$\tilde{W}=u_0^{\alpha_0}\tilde{x}_1u_1^{\alpha_1}\tilde{x}_2\ldots u_{n-1}^{\alpha_{n-1}}\tilde{x}_nu_n^{\alpha_n},\quad \alpha_i=\alpha_{i-1}\sigma(x_i),\ \tilde{x}_i=u_{i-1}^{\alpha_{i-1}}u_i^{\alpha_i}. \tag{8}$$

A lift $\tilde{W}$ is call a *resolution* of W if it is a cycle or an open path. There are exactly two lifts of W in the form (8), since there are exactly two choices for α_0. Moreover, it follows from (5) that the lifted orientation $\tilde{\varepsilon}_W$ from ε_W by (2) to the edges in $\tilde{W}$ forms a direction of $\tilde{W}$. Thus (W,ε_W) is lifted exactly to two directed walks $(\tilde{W},\tilde{\varepsilon}_W)$. We call W the *projected walk* of $\tilde{W}$ and write $W=\pi(\tilde{W})$.

Lemma 2. *Let W be a closed walk in Σ with initial and terminal vertices, and let ε_W be a direction of W with respect to the initial vertex. Let $(\tilde{W}, \tilde{\varepsilon}_W)$ be a lift of the directed walk (W, ε_W). If W is positive, then $(\tilde{W}, \tilde{\varepsilon}_W)$ is a directed closed walk. If W is negative, then $(\tilde{W}, \tilde{\varepsilon}_W)$ is a directed open walk.*

Proof. Let $W = u_0 x_1 u_1 x_2 \ldots u_{n-1} x_n u_n$ and be lifted to a walk $\tilde{W} = u_0^{\alpha_0} \tilde{x}_1 u_1^{\alpha_1} \ldots \tilde{x}_n u_n^{\alpha_n}$. Since positive edges are lifted to the edges staying in the same level and negative edges to the edges crossing the two levels, we see that $\sigma(x_i) = \alpha_{i-1}\alpha_i$. Then

$$\sigma(W) = \prod_{i=1}^{n} \sigma(x_i) = \prod_{i=1}^{n} \alpha_{i-1}\alpha_i = \alpha_0 \alpha_n.$$

Clearly, $\sigma(W) = +1$ if and only if $\alpha_n = \alpha_0$, and subsequently if and only if $\tilde{W}$ is closed. $\square$

2. Flows

Flows on signed graphs. The *incidence matrix* of an oriented signed graph (Σ, ε) is the matrix $\boldsymbol{M} = \boldsymbol{M}(\Sigma, \varepsilon) = [\boldsymbol{m}(u, x)]$ indexed by the set $V \times E$, where $\boldsymbol{m}$ is a function $\boldsymbol{m} : V \times E \to \mathbb{Z}$ defined by

$$\boldsymbol{m}(u, x) = \begin{cases} \varepsilon(u, x) & \text{if } x \text{ is a link,} \\ 2\varepsilon(u, x) & \text{if } x \text{ is a negative loop,} \\ 0 & \text{otherwise.} \end{cases}$$

Alternatively, $\boldsymbol{m}(u, x) = \sum_{v \in \mathrm{End}(x),\, v = u} \varepsilon(v, x)$, where $\mathrm{End}(x)$ is a multiset of two identical vertices when x is a loop. An *integral flow* on an oriented signed graph (Σ, ε) is a function $f : E \to \mathbb{Z}$ which is *conservative* at every vertex u, meaning that $\partial f = 0$, where $\partial : \mathbb{Z}^E \to \mathbb{Z}^V$ is called the *boundary operator* of (Σ, ε), defined by

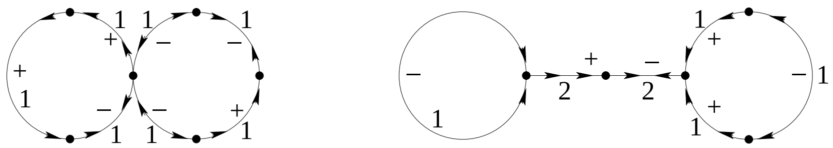
$$\partial f(u) = \sum_{x \in E} \boldsymbol{m}(u, x) f(x) = \sum_{x \in E} \sum_{v \in \mathrm{End}(x),\, v = u} \varepsilon(u, x) f(x). \tag{9}$$

The value of the function ∂f at u is also called the *excess* of f at u in network flows. The set of all integral flows on (Σ, ε) forms a $\mathbb{Z}$-module, called the *flow lattice* by Chen and Wang, who developed its basic theory in [5]. One can define flows with values in an arbitrary abelian group, such as the additive reals and finitely generated abelian groups. Many of the following remarks are applicable in general, so we omit the word "integral."

The theory of flows of signed graphs depends essentially on the graph and sign function but not on the orientation, since for two orientations ε, ρ there is an isomorphism from the flow lattice of (Σ, ε) to the flow lattice of (Σ, ρ) by $f \mapsto [\varepsilon, \rho] f$.

The *support* of a function $f : E \to \mathbb{Z}$ is the set of edges x such that $f(x) \neq 0$, denoted $\operatorname{supp} f$. We denote by $\Sigma(f)$ the signed subgraph of Σ whose edge set is $\operatorname{supp} f$ and vertex set consists of vertices incident with some edges in $\operatorname{supp} f$. The flow that is zero on all edges is called the *zero flow*. Flows other than the zero flow are referred to *nontrivial flows*. A *circuit flow* is a flow whose support is a signed-graph circuit, having values ± 1 on the edges of the cycles and value ± 2 on the edges of the circuit path (for Type III circuits). See Figure 3 for signed-graph circuit flows of Type II and Type III.

The theory of flows on ordinary graphs is simply the case that all edges are positive, where a circuit flow has a cycle as its support and has values ± 1 on the edges of the cycle (see [2],



(a) Type II (b) Type III

Figure 3. Signed-graph circuit flows of Types II and III

p.52). Signed-graph circuit flows are defined analogously in [5]. Nowhere-zero integral flows on signed graphs are studied in [1, 4, 11].

We say that an integral flow f_1 *conforms to the sign pattern of* f if $\operatorname{supp} f_1 \subseteq \operatorname{supp} f$, and $f_1(x)$ has the same sign as $f(x)$ for all edges x in $\operatorname{supp} f_1$.

An integral flow f on (Σ, ε) lifts to a flow on the oriented sign-labeled covering graph $(\tilde{\Sigma}, \tilde{\varepsilon})$, possibly in more than one way. The best way to see this is through the correspondence between flows and walks, which exists if $\Sigma(f)$ is connected.

A directed closed positive walk (W, ε_W) on (Σ, ε) corresponds to a unique integral *flow* $f_{(W,\varepsilon_W)}$, defined by

$$f_{(W,\varepsilon_W)}(x) = \sum_{y \in W,\, y=x} [\varepsilon, \varepsilon_W](y), \tag{10}$$

where W is viewed as a multiset $\{x_1, x_2, \ldots, x_n\}$ of edges if $W = u_0x_1u_1 \ldots x_nu_n$; see [5]. Clearly, $f_{(W,\varepsilon_W)}(x)$ is the number of times that the walk W traverses the edge x along the direction ε minus the number of times that W traverses x along the direction opposite to ε. Whenever the direction ε_W is the same as ε on W, we simply write $f_{(W,\varepsilon_W)}$ as f_W, and

$$f_W(x) = \#\{y \in W \mid y = x\} \text{ (as a multiset)}. \tag{11}$$

To see that $f_{(W,\varepsilon_W)}$ is a flow, consider the contribution to $f_{(W,\varepsilon_W)}$ of a pair of consecutive edges, $x_iu_ix_{i+1}$, at the intervening vertex u_i. Since (W, ε_W) is coherent at u_i, the contribution of these edges to $\partial f_{(W,\varepsilon_W)}(u_i)$ is 0. This same argument applies to the initial vertex if we take subscripts modulo the length of W. We can apply the same definition $f_{(W,\varepsilon_W)}$ to any directed walk (W, ε_W) (not necessarily closed), however the result is no longer a flow. In fact, we have the following lemma.

Lemma 3. *Let* (W, ε_W) *be a directed walk with* $W = u_0x_1u_1x_2 \ldots u_{n-1}x_nu_n$. *Then the function* $f_{(W,\varepsilon_W)}$ *is conservative everywhere except at* u_0 *and* u_n, *that is,*

$$\partial f_{(W,\varepsilon_W)}(u) = 0, \quad u \neq u_0, u_n.$$

Moreover, if W *is closed, then*

$$\partial f_{(W,\varepsilon_W)}(u_0) = \begin{cases} 0 & \text{if } W \text{ is positive,} \\ 2\varepsilon_W(u_0, x_1) & \text{if } W \text{ is negative.} \end{cases}$$

If W *is open, then* $\partial f_{(W,\varepsilon_W)}(u_0) = \varepsilon_W(u_0, x_1)$ *and*

$$\partial f_{(W,\varepsilon_W)}(u_n) = \varepsilon_W(u_n, x_n) = -\sigma(W)\, \varepsilon_W(u_0, x_1).$$

Proof. Fix a vertex u. Let u be appeared in the vertex-edge sequence of W as the subsequence $u_{\ell_1}, u_{\ell_2}, \ldots, u_{\ell_k}$. If $u \neq u_0, u_n$, then $\varepsilon_W(u_{\ell_i}, x_{\ell_i}) = -\varepsilon_W(u_{\ell_i}, x_{\ell_i+1})$. We compute

$$\begin{aligned}
\partial f_{(W,\varepsilon_W)}(u) &= \sum_{x\in E} \boldsymbol{m}(u,x) \sum_{y\in W,\, y=x} [\varepsilon, \varepsilon_W](y) \\
&= \sum_{y\in W} \boldsymbol{m}(u,y)[\varepsilon, \varepsilon_W](y) \\
&= \sum_{y\in W} \sum_{v\in \mathrm{End}(y),\, v=u} \varepsilon(v,y)[\varepsilon, \varepsilon_W](y) \\
&= \sum_{y\in W} \sum_{v\in \mathrm{End}(y),\, v=u} \varepsilon_W(v,y).
\end{aligned}$$

The last equality follows from definition of the coupling. For each vertex u_{ℓ_i} with $1 \leq i \leq k$, we have the sub-walk segment $x_{\ell_i} u_{\ell_i} x_{\ell_i+1}$, which contributes exactly the two terms $\varepsilon_W(u_{\ell_i}, x_{\ell_i})$, $\varepsilon_W(u_{\ell_i}, x_{\ell_i+1})$ in the above last sum. Thus

$$\partial f_{(W,\varepsilon_W)}(u) = \sum_{i=1}^{k} [\varepsilon_{(W,\varepsilon_W)}(u_{\ell_i}, x_{\ell_i}) + \varepsilon_W(u_{\ell_i}, x_{\ell_i+1})] = 0$$

regardless of whether W is closed or open.

Assume W is closed, that is, $u_0 = u_n$. If $u = u_0$, then $\ell_1 = u_0$ and $\ell_k = n$. So u_{ℓ_1} and u_{ℓ_k} are the initial and terminal vertices of W, $x_0 = x_n$ and $x_{n+1} = x_1$. We have

$$\partial f_{(W,\varepsilon_W)}(u_0) = \varepsilon_W(u_0, x_1) + \varepsilon(u_n, x_n).$$

In particular, if W is positive, we have $\partial f_{(W,\varepsilon_W)}(u_0) = 0$; if W is negative, we have

$$\partial f_{(W,\varepsilon_W)}(u_0) = 2\varepsilon(u_0, x_1) = 2\varepsilon_W(u_n, x_n).$$

Assume W is open, that is, $u_0 \neq u_n$. If $u = u_0$ or $u = u_n$, then $\partial f_{(W,\varepsilon_W)}(u_0) = \varepsilon_W(u_0, x_1)$ and

$$\partial f_{(W,\varepsilon_W)}(u_n) = \varepsilon_W(u_n, x_n) = -\sigma(W)\varepsilon_W(u_0, x_1).$$

□

Conversely, directed closed walks (which are usually not unique) can be constructed from integral flows. Chen and Wang [6] developed a *flow reduction algorithm* to obtain an equivalent classification of indecomposable flows. The method employed in [6] is an algorithmic approach. The present paper is a structural approach by resolution of singularities.

Proposition 4. *Let f be a nonnegative, nontrivial, integral flow on (Σ, ε) such that $\Sigma(f)$ is connected. Then there exists a directed, closed, positive walk (W, ε) such that $f_W = f$.*

Proof. We apply induction on the total weight $\|f\| := \sum_{x\in E} f(x)$. Choose a vertex u_0 and an edge x_1 incident with u_0 in $\Sigma(f)$. Let u_1 be the other end-vertex of x_1 ($u_1 = u_0$ if x_1 is a loop). This gives a walk $W_1 = u_0 x_1 u_1$ of length 1. Clearly, $f \geq f_{W_1} \geq 0$.

Now assume that we have selected a partial walk $W_k = u_0 x_1 u_1 \ldots x_k u_k$ and $f \geq f_{W_k} \geq 0$. If W_k is not a closed positive walk, that is, W is open, or closed but is negative, then by Lemma 3, the function f_{W_k} is not a flow, for it is not conservative at u_k. Since f is conservative at u_k, the function $f - f_{W_k}$ cannot be conservative at u_k. Then there exists an edge x_{k+1} incident with u_k in $\Sigma(f - f_{W_k})$ such that $\varepsilon(u_k, x_{k+1}) = -\varepsilon(u_k, x_k)$. Let u_{k+1} denote the other end-vertex of x_{k+1} and extend W_k to

$$W_{k+1} = W_k x_{k+1} u_{k+1}.$$

Clearly, we have $f \geq f_{W_{k+1}} \geq 0$ by the construction. Continue this procedure when W_{k+1} is not a closed positive walk. We finally obtain a directed closed positive walk (W_n, ε). Thus f' $(= f - f_{W_n})$ is a nonnegative integral flow and clearly $\|f'\| < \|f\|$.

Let $\Sigma(f')$ be decomposed into connected components Σ_i, and set $f'_i = f'|_{\Sigma_i}$. Then $f' = \sum_i f'_i$ and $\operatorname{supp} f'_i = E(\Sigma_i)$. It is easy to see that f'_i are nonnegative nontrivial integral flows of (Σ, ε) and $\|f'_i\| < \|f\|$. By induction, there exist directed closed positive walks (W'_i, ε) such that $f'_i = f_{W'_i}$. It is clear that the union of all (W'_i, ε) and (W_n, ε) is connected. One can construct a single directed closed positive walk (W, ε) by rearranging the initial and terminal vertices of all W'_i and W_n, and connecting them properly at some of their intersections. Clearly, we have $f_W = f$ by the construction. $\square$

Let f be an integral flow of (Σ, ε). Associated with f is an orientation ε_f on Σ defined by (for each edge x at its end-vertex u)

$$\varepsilon_f(u, x) = \begin{cases} \varepsilon(u, x) & \text{if } f(x) \geq 0, \\ -\varepsilon(u, x) & \text{if } f(x) < 0. \end{cases} \tag{12}$$

Corollary 5. *Let f be a nontrivial integral flow on (Σ, ε). If $\Sigma(f)$ is connected, then there exists a closed positive walk W such that (W, ε_f) is a directed closed walk and $f_{(W,\varepsilon_f)} = f$.*

Proof. The absolute function $|f| = [\varepsilon, \varepsilon_f] f$ is a nonnegative, nontrivial, integral flow of (Σ, ε_f). According to Proposition 4, there exists a directed closed positive walk (W, ε_f) such that $f_W = |f|$ within the oriented signed graph (Σ, ε_f), where

$$f_W(x) = \sum_{y \in W,\, y = x} [\varepsilon_f, \varepsilon_f](y) = \sum_{y \in W,\, y = x} 1, \quad x \in E.$$

For the same directed closed positive walk (W, ε_f) within (Σ, ε), we have

$$\begin{aligned} f_{(W,\varepsilon_f)}(x) &= \sum_{y \in W,\, y = x} [\varepsilon, \varepsilon_f](y) = [\varepsilon, \varepsilon_f](x) \sum_{y \in W,\, y = x} 1 \\ &= [\varepsilon, \varepsilon_f](x) f_W(x), \quad x \in E. \end{aligned}$$

Since $f = [\varepsilon, \varepsilon_f]\, |f|$ and $|f| = f_W$, it follows that $f_{(W,\varepsilon_f)} = f$. $\square$

Lifted flows. Consider a function $\tilde{f} : \tilde{E} \to \mathbb{Z}$ defined on the edge set of the sign-labeled covering graph $\tilde{\Sigma}$. The *projection* of $\tilde{f}$ is the function $\pi(\tilde{f}) : E \to \mathbb{Z}$ defined by

$$\pi(\tilde{f})(x) = \tilde{f}(x^+) + \tilde{f}(x^-). \tag{13}$$

Let $\tilde{\boldsymbol{M}} = \boldsymbol{M}(\tilde{\Sigma}, \tilde{\varepsilon})$ denote the incidence matrix of $(\tilde{\Sigma}, \tilde{\varepsilon})$ of size $\tilde{V} \times \tilde{E}$. Write $\tilde{\boldsymbol{M}} = [\tilde{\boldsymbol{m}}(u^\alpha, \tilde{x})]$. Then $\tilde{\boldsymbol{m}}(u^\alpha, \tilde{x}) = \tilde{\varepsilon}(u^\alpha, \tilde{x})$ if $\tilde{x}$ is a link edge with end-vertex u^α, and $\tilde{\boldsymbol{m}}(u^\alpha, \tilde{x}) = 0$ otherwise. Since only positive loops of Σ are lifted to loops in $\tilde{\Sigma}$, we have

$$\tilde{\boldsymbol{m}}(u^\alpha, \tilde{x}) = \begin{cases} 0 & \text{if } x \text{ is a positive loop}, \\ \alpha\, \varepsilon(u, x) & \text{otherwise}. \end{cases} \tag{14}$$

The boundary operator $\partial : \mathbb{Z}^{\tilde{E}} \to \mathbb{Z}^{\tilde{V}}$ is defined by

$$\partial \tilde{f}(u^\alpha) = \sum_{y \in \tilde{E}} \tilde{\boldsymbol{m}}(u^\alpha, y)\tilde{f}(y).$$

Lemma 6. *Let $\tilde{f}$ be a function defined on $\tilde{E}$. Then $\partial\pi(\tilde{f})$ is given by*

$$\partial\pi(\tilde{f})(u) = \partial\tilde{f}(u^+) - \partial\tilde{f}(u^-), \quad u \in V. \tag{15}$$

If $\tilde{f}$ is a flow on $(\tilde{\Sigma}, \tilde{\varepsilon})$, so is $\pi(\tilde{f})$ on (Σ, ε).

Proof. Fix a vertex u in Σ. Let E_u denote the set of all edges incident with u. Then E_u is partitioned into the disjoint sets $\mathrm{Lk}(u), \mathrm{Lp}(u)$ of links and loops at u respectively, and $\mathrm{Lp}(u)$ is further partitioned into the disjoint subsets

$$\mathrm{Lp}_+(u) = \{\text{positive loops at } u\}, \quad \mathrm{Lp}_-(u) = \{\text{negative loops at } u\}.$$

Since $\boldsymbol{m}(u, x) = 0$ if x is a positive loop at u, we compute

$$\begin{aligned} \partial\pi(\tilde{f})(u) &= \sum_{x \in E_u} \boldsymbol{m}(u, x)\big[\tilde{f}(x^+) + \tilde{f}(x^-)\big] \\ &= \sum_{x \in \mathrm{Lk}(u)} \varepsilon(u, x)[\tilde{f}(x^+) + \tilde{f}(x^-)] \qquad (16) \\ &\quad + \sum_{x \in \mathrm{Lp}_-(u)} 2\varepsilon(u, x)[\tilde{f}(x^+) + \tilde{f}(x^-)]. \qquad (17) \end{aligned}$$

Fix a vertex u^α of $\tilde{\Sigma}$. The link edges incident with u^α are the edges x^α with $x \in \mathrm{Lk}(u)$ and the edges $x^\alpha, x^{-\alpha}$ with $x \in \mathrm{Lp}_-(u)$. We compute

$$\begin{aligned} \partial\tilde{f}(u^\alpha) &= \sum_{x \in \mathrm{Lk}(u) \cup \mathrm{Lp}_-(u)} \tilde{\boldsymbol{m}}(u^\alpha, x^\alpha)\tilde{f}(x^\alpha) + \sum_{x \in \mathrm{Lp}_-(u)} \tilde{\boldsymbol{m}}(u^\alpha, x^{-\alpha})\tilde{f}(x^{-\alpha}) \\ &= \alpha \sum_{x \in \mathrm{Lk}(u) \cup \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^\alpha) + \alpha \sum_{x \in \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^{-\alpha}). \end{aligned}$$

The second equality follows from (14). In particular, we have

$$\begin{aligned} \partial\tilde{f}(u^+) &= \sum_{x \in \mathrm{Lk}(u) \cup \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^+) + \sum_{x \in \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^-); \\ \partial\tilde{f}(u^-) &= -\sum_{x \in \mathrm{Lk}(u) \cup \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^-) - \sum_{x \in \mathrm{Lp}_-(u)} \varepsilon(u, x)\tilde{f}(x^+). \end{aligned}$$

Adding $\partial\tilde{f}(u^+)$ and $-\partial\tilde{f}(u^-)$ together and comparing the result with $\partial\pi(\tilde{f})(u)$ in (16), we obtain (15).

Whenever $\tilde{f}$ is a flow of $(\tilde{\Sigma}, \tilde{\varepsilon})$, then $\partial\tilde{f}(u^+) = \partial\tilde{f}(u^-) = 0$. Thus $\partial\pi(\tilde{f})(u) = 0$. This means that $\partial\pi(\tilde{f})$ is a flow of (Σ, ε). □

A *lift* of an integral flow f of (Σ, ε) to $\tilde{\Sigma}$ is an integral flow $\tilde{f}$ of $(\tilde{\Sigma}, \tilde{\varepsilon})$ such that $\pi(\tilde{f}) = f$.

Proposition 7. *(a) Let $(\tilde{W}, \tilde{\varepsilon}_W)$ be a lift of a directed closed positive walk (W, ε_W). Then*

$$\pi[f_{(\tilde{W}, \tilde{\varepsilon}_W)}] = f_{(W, \varepsilon_W)}. \tag{18}$$

(b) Let f be an integral flow on (Σ, ε), and let (W, ε_f) be a directed closed positive walk such that $f = f_{(W,\varepsilon_f)}$. Let (W, ε_f) be lifted to a directed closed walk $(\tilde{W}, \tilde{\varepsilon}_f)$ in $\tilde{\Sigma}$. Then $\tilde{f} := f_{(\tilde{W},\tilde{\varepsilon}_f)}$ is a flow on $(\tilde{\Sigma}, \tilde{\varepsilon})$ lifted from f. Moreover, if f is nonnegative, so is $\tilde{f}$.

Proof. (a) It suffices to show that $f_{(W,\varepsilon_W)}(x) = f_{(\tilde{W},\tilde{\varepsilon}_W)}(x^+) + f_{(\tilde{W},\tilde{\varepsilon}_W)}(x^-)$. Since the lifting of orientations preserves the coupling, then by (7) we have

$$\begin{aligned} \pi[f_{(\tilde{W},\tilde{\varepsilon}_W)}](x) &= f_{(\tilde{W},\tilde{\varepsilon}_W)}(x^+) + f_{(\tilde{W},\tilde{\varepsilon}_W)}(x^-) \\ &= \sum_{\tilde{y}\in\tilde{W},\ \tilde{y}=x^+ \text{ or } x^-} [\tilde{\varepsilon}, \tilde{\varepsilon}_W](\tilde{y}) \\ &= \sum_{y\in W,\ y=x} [\varepsilon, \varepsilon_W](y) = f_{(W,\varepsilon_W)}(x). \end{aligned}$$

(b) Since $f = f_{(W,\varepsilon_f)}$, it is trivial by (18) that $\tilde{f}$ is a lifted flow of f. If f is nonnegative, then $\varepsilon_f = \varepsilon$; subsequently, $\tilde{\varepsilon}_f = \tilde{\varepsilon}$. Hence $\tilde{f} = f_{(\tilde{W},\tilde{\varepsilon}_f)}$ is nonnegative by definition (10). ☐

An integral flow f on an oriented signed graph (Σ, ε) is said to be *conformally decomposable* if it is nontrivial and can be represented as a sum of two other integral flows, $f = f_1 + f_2$, each of which is nontrivial and conforms to the sign pattern of f, that is, $f_1(x)f_2(x) \geq 0$ for all edges x; this means that both $f_1(x), f_2(x)$ are positive or both are negative if they are nonzero. An integral flow of (Σ, ε) is said to be *conformally indecomposable* if it is not conformally decomposable. A nonnegative nontrivial integral flow f on (Σ, ε) is said to be *minimal* provided that if g is a nonnegative nontrivial integral flow on (Σ, ε) such that $g(x) \leq f(x)$ for all edges x, then $g = f$.

If an integral flow f is nonnegative, then minimality is equivalent to conformal indecomposability. In fact, if f is decomposed into $f = f_1 + f_2$, then f_1, f_2 must be nonnegative nontrivial integral flows and $f_1 \leq f$, $f_1 \neq f$; this means that f is not minimal. Conversely, if f is not minimal, say, there is a nonnegative nontrivial integral flow g on (Σ, ε) such that $g \leq f$ but $g \neq f$, then $h = f - g$ is nontrivial and nonnegative, and f is decomposed into $f = g + h$.

In the following we shall see that the conformal indecomposability of a nontrivial integral flow f on (Σ, ε) is equivalent to the *minimality* of the absolute value flow $|f|$ on (Σ, ε_f), where ε_f is the orientation given by (12). It is well known and easy to see that conformally indecomposable flows on an ordinary graph are just graph circuit flows.

Lemma 8. *A nontrivial integral flow f of (Σ, ε) is conformally indecomposable if and only if the absolute value function $|f|$ is a minimal flow of (Σ, ε_f).*

Proof. Applying the boundary operator (9), it is clear that f is a flow on (Σ, ε) if and only if $|f| = [\varepsilon, \varepsilon_f]\, f$ is a flow on (Σ, ε_f). Since $|f|$ is nonnegative, its minimality is equivalent to its conformal indecomposability. We show necessity first. Suppose $|f|$ is not minimal, that is, $|f| = g_1 + g_2$, where g_i are nonnegative nontrivial integral flows of (Σ, ε_f). Then $f_i = [\varepsilon, \varepsilon_f]\, g_i$ are nontrivial integral flows of (Σ, ε). Thus

$$f = [\varepsilon, \varepsilon_f]\, |f| = [\varepsilon, \varepsilon_f]\, g_1 + [\varepsilon, \varepsilon_f]\, g_2 = f_1 + f_2,$$

and $f_1\, f_2 = g_1\, g_2 \geq 0$, meaning that f is conformally decomposable. This is a contradiction.

For sufficiency, suppose f is conformally decomposable, that is, $f = f_1 + f_2$, where f_i are nontrivial integral flows of (Σ, ε) such that $f_1\, f_2 \geq 0$. Then $g_i = [\varepsilon, \varepsilon_f]\, f_i$ are nontrivial integral flows of (Σ, ε_f). For each edge x, if $f_i(x) > 0$, we must have $f(x) > 0$ and $[\varepsilon, \varepsilon_f](x) = 1$; if $f_i(x) < 0$, we must have $f(x) < 0$ and $[\varepsilon, \varepsilon_f](x) = -1$; then $g_i(x) \geq 0$. Hence

$$|f| = [\varepsilon, \varepsilon_f]\, f = [\varepsilon, \varepsilon_f]\, f_1 + [\varepsilon, \varepsilon_f]\, f_2 = g_1 + g_2,$$

meaning that $|f|$ is conformally decomposable. This is a contradiction. □

3. Indecomposable flows

A signed graph Ω with nonempty edge set is said to be *Eulerian* if there exists a directed closed walk (W, ε_W) that uses every edge of Ω at least once but at most twice, and the direction ε_W has the same orientation on each pair of repeated edges. An Eulerian signed graph is further said to be *prime* if its directed closed walk does not properly contain directed closed subwalks. An Eulerian signed graph is said to be *mimimal* if it does not properly contain Eulerian signed subgraphs. It is easy to see that minimal Eulerian signed graphs must be prime Eulerian signed graphs.

Definition 9. *A signed graph T with nonempty edge set is called a* cycle-tree *if it satisfies the following conditions:*

(a) *T is connected.*
(b) *Each block of T is either a cycle (called* block cycle*) or an edge.*
(c) *Each cut-vertex is incident with exactly two blocks.*
(d) *Each block incident with exactly one cut-vertex is a cycle (called an* end-block cycle*).*

A cycle-tree is said to be Eulerian *if it further satisfies*

(e) Parity Condition: *The sign of a block cycle equals $(-1)^p$, where p is the number of cut-vertices of T on the cycle.*

We shall see that prime Eulerian signed graphs are Eulerian cycle-trees, and minimal Eulerian signed graphs are signed-graph circuits, i.e., positive cycles or handcuffs.

A cycle-tree is indeed a tree-like signed graph whose "vertices" are the block cycles and "edges" are the paths (of possible zero length, called *block paths*) between pairs of block cycles. The end-vertices of block paths are cut-vertices. If a block path has length zero, it is a common cut-vertex of two block cycles. We may also think of an Eulerian cycle-tree as a "tree" whose "vertices" are some vertex-disjoint maximal Eulerian subgraphs and "edges" are the paths (of positive length) between the Eulerian subgraphs, where each such maximal Eulerian graph is also a tree-like whose "vertices" are edge-disjoint cycles and "edges" are the intersection vertices between pairs of cycles. Figure 4 exhibits an Eulerian cycle-tree with a direction.

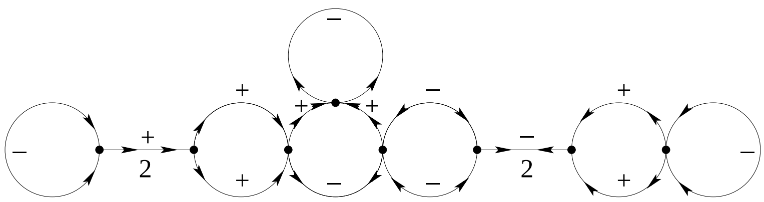


Figure 4. An Eulerian cycle-tree with a direction

Let T be an Eulerian cycle-tree. A *direction* of T is an orientation ε_T on T such that (T, ε_T) has neither sources nor sinks, but for each block cycle C the signed subgraph (C, ε_T)

has either a source or a sink at each cut-vertex of T on C. It is easy to see that there exist exactly two (opposite) directions on T. For unsigned graphs, since all edges are positive, Eulerian cycle-trees are just cycles, and directed Eulerian cycle-trees are directed cycles.

When T is contained in a signed graph Σ, the *indicator function* of T is the function $I_T : E \to \mathbb{Z}$ defined by

$$I_T(x) = \begin{cases} 1 & \text{if } x \text{ belongs to a block cycle,} \\ 2 & \text{if } x \text{ belongs to a block path,} \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

The function $[\varepsilon, \varepsilon_T]\, I_T$ is called the *characteristic vector* of (T, ε_T) within (Σ, ε).

A closed walk on T is called an *Eulerian tour* if it uses every edge of T and has minimum length. A subgraph of T is called an *Eulerian cycle-subtree* if it is also an Eulerian cycle-tree and its block cycles and block paths are block cycles and block paths of T.

Proposition 10 (Existence and Uniqueness of Direction on Eulerian Cycle-Tree)**.** *Let T be an Eulerian cycle-tree. Then*

(a) *There exists a closed walk W on T such that (i) W uses each edge of block cycles once and each edge of block paths twice, (ii) whenever W meets a cut-vertex, it crosses from one block to another block.*

Moreover, each such closed walk is an Eulerian tour on T, having the length

$$\ell(T) := \sum_i \ell(C_i) + 2 \sum_j \ell(P_j), \tag{20}$$

where C_i are block cycles and P_j are block paths of T.

(b) *Each Eulerian tour W on T satisfies the conditions (i), (ii), and $\ell(W) = \ell(T)$.*

(c) *There exists a unique direction ε_T of T (up to opposite sign) such that (W, ε_T) is coherent for each Eulerian tour W on T and*

$$f_{(W,\varepsilon_T)} = [\varepsilon, \varepsilon_T]\, I_T. \tag{21}$$

Moreover, if u is a cut-vertex and $W = W_1 W_2$, where W_i are closed sub-walks having initial and terminal vertices at u, then both W_i are negative and (W_i, ε_T) are incoherent at u and coherent elsewhere.

Proof. (a) & (b): Let $\tilde{T}$ be the sign-labeled covering graph of T. We claim that there exists a directed cycle $(\tilde{W}, \varepsilon_{\tilde{W}})$ such that (i′) $\tilde{W}$ covers each edge of block cycles once and each edge of block paths twice, (ii′) orientations on each edge of T induced from $(\tilde{W}, \varepsilon_{\tilde{W}})$ by the projection are identical, and (iii′) the induced orientation ε_T from $\varepsilon_{\tilde{W}}$ is a direction of T.

We proceed by induction on the number of block cycles. If there is only one block cycle, the cycle must be positive as there is no cut vertices. Choose a direction of the cycle. The directed cycle lifts to a directed cycle in $\tilde{T}$ by Lemma 2. Now we assume that T contains at least two block cycles.

Choose an end-block cycle C_0 of T with the unique cut-vertex u_0. There exists a path P connecting C_0 to another block cycle C_1, having the initial vertex u_0 on C_0 and the terminal vertex v_0 on C_1. Let C_0, P be written as

$$C_0 = u_0 x_1 u_1 x_2 \ldots u_{m-1} x_m u_m, \quad P = v_0 y_1 v_1 y_2 \ldots v_{n-1} y_n v_n,$$

where $u_0 = u_m = v_n$, and be lifted to $\tilde{T}$ as the paths

$$\tilde{P}_0 = u_0^{\alpha_0} \tilde{x}_1 u_1^{\alpha_1} \tilde{x}_2 \ldots u_{m-1}^{\alpha_{m-1}} \tilde{x}_m u_m^{\alpha_m}, \quad \tilde{P} = v_0^{\beta_0} \tilde{y}_1 v_1^{\beta_1} \tilde{y}_2 \ldots v_{n-1}^{\beta_{n-1}} \tilde{y}_n v_n^{\beta_n}.$$

Note that $\alpha_m = -\alpha_0$, since the cycle C_0 is negative.

Let us remove C_0, P from T, change the sign of an edge z in C_1 at v_0, and rename z as z_1. We then obtain an Eulerian cycle-tree T_1, which has one fewer block cycles than T. By induction there exists a directed cycle $(\tilde{W}_1, \tilde{\varepsilon}_{W_1})$ satisfying the conditions (i′)–(iii′). Let $\tilde{z}_1$ be an edge of $\tilde{W}_1$ that covers z_1, having an end-vertex v_0^{γ} covering v_0. Let us write $\tilde{W}_1$ as a closed path from v_0^{γ} to v_0^{γ}, having arranged $\tilde{z}_1$ as the last edge. Let $s = \tilde{\varepsilon}_{W_1}(v_0^{\gamma}, \tilde{z}_1)$, and let $(\tilde{P}_1, \tilde{\varepsilon}_{P_1})$ be a directed path in $\tilde{T}$ from v_0^{γ} to $v_0^{-\gamma}$, obtained from $(\tilde{W}_1, \tilde{\varepsilon}_{W_1})$ by replacing the edge $\tilde{z}_1$ with an edge $\tilde{z}$ of $\tilde{T}$ at $v_0^{-\gamma}$, where $\tilde{z}$ covers z, having the orientation $\tilde{\varepsilon}_{P_1}(v_0^{-\gamma}, \tilde{z}) = s$.

In the case that P has length zero, we have $u_0 = v_0$. Let $\alpha_0 = -\gamma$. Then $\tilde{P}_0$ is an open path from $u_0^{-\gamma}$ to u_0^{γ}. Choose a direction $\tilde{\varepsilon}_{P_0}$ of $\tilde{P}_0$ such that $\tilde{\varepsilon}_{P_0}(u_0^{\alpha_0}, \tilde{x}_1) = -s$. Then $(\tilde{W}, \varepsilon_{\tilde{W}})$ is a directed cycle in $\tilde{T}$, satisfying the conditions (i′)–(iii′), where $\tilde{W} = \tilde{P}_0\tilde{P}_1$ and $\varepsilon_{\tilde{W}} = \tilde{\varepsilon}_{P_0} \vee \tilde{\varepsilon}_{P_1}$.

In the case that P has positive length, let $\beta_0 = -\gamma$. Then β_n is determined by $\tilde{P}$. Let $\alpha_0 = \beta_n$. Then $\tilde{P}\tilde{P}_0$ is an open path from $v_0^{-\gamma}$ to $u_0^{-\alpha_0}$. Choose a direction $\tilde{\varepsilon}_P$ of $\tilde{P}$ such that $\tilde{\varepsilon}_P(v_0^{\beta_0}, \tilde{y}_1) = -s$, then $\tilde{\varepsilon}_P(v_n^{\beta_n}, \tilde{y}_n) = s$. Choose a direction $\tilde{\varepsilon}_{P_0}$ of $\tilde{P}_0$ such that $\tilde{\varepsilon}_{P_0}(u_0^{\alpha_0}, \tilde{x}_1) = -s$, then $\tilde{\varepsilon}_{P_0}(u_m^{\alpha_m}, \tilde{x}_m) = s$. Consider the path

$$\tilde{P}^* = v_0^{-\beta_0}\tilde{y}_1^* v_1^{-\beta_1}\tilde{y}_2^* \ldots v_{n-1}^{-\beta_{n-1}}\tilde{y}_n^* v_n^{-\beta_n};$$

its reverse $\tilde{P^*}^{-1}$ is a path from $u_0^{\alpha_m}$ to v_0^{γ}. There is an induced direction $\tilde{\varepsilon}_P^*$ on $\tilde{P}^*$ by the augmentation *, that is, $\tilde{\varepsilon}_P^*(v_0^{-\beta_0}, \tilde{y}_1^*) = s$, $\tilde{\varepsilon}_P^*(v_n^{-\beta_n}, \tilde{y}_n^*) = -s$. Note that the direction $\tilde{\varepsilon}_P$ of $\tilde{P}$ induces a direction ε_P on P by the projection, that is, $\varepsilon_P(v_0, y_1) = -\beta_0 s$, $\varepsilon_P(v_0, y_n) = \beta_n s$. The direction $\tilde{\varepsilon}_P^*$ of $\tilde{P}^*$ also induces the direction ε_P by the projection. Thus $(\tilde{W}, \varepsilon_{\tilde{W}})$ is a directed cycle in $\tilde{T}$, satisfying the conditions (i′)–(iii′), where $\tilde{W} = \tilde{P}\tilde{P}_0\tilde{P^*}^{-1}\tilde{P}_1$ and $\varepsilon_{\tilde{W}} = \tilde{\varepsilon}_P \vee \tilde{\varepsilon}_{P_0} \vee \tilde{\varepsilon}_P^* \vee \tilde{\varepsilon}_{P_1}$.

We define the closed walk $W = \pi(\tilde{W})$ and its direction $\varepsilon_W = \pi(\varepsilon_{\tilde{W}})$. Then W satisfies the conditions (i), (ii), and is an Eulerian tour on T.

Conversely, each Eulerian tour W on T must pass each edge of block paths at least twice, for each edge of block paths is a cut-edge. So W has length at least $\ell(T)$. Since W has minimum length, it follows that $\ell(W) = \ell(T)$. The minimum property of $\ell(W)$ forces that W satisfies the conditions (i) and (ii).

(c) Let ε_T be the direction of T by taking the projection of the direction $\varepsilon_{\tilde{W}}$. Now for an arbitrary Eulerian tour W on T, the direction property of ε_T forces (W, ε) to be a directed closed walk. The uniqueness of ε_T up to opposite sign follows from the "tree" structure of T. The properties (i) and (ii) imply that I_T is a flow of (T, ε_T). Thus $f_{(W,\varepsilon_T)} = [\varepsilon, \varepsilon_T]\, I_T$. $\square$

It is easy to see that the number of Eulerian tours (up to the reversing of orders of vertex-edge sequences) on an Eulerian cycle-tree T is 2^q, where q is the number of cut-vertices of T.

Lemma 11 (Minimality of Eulerian Cycle-Tree). *Let T be an Eulerian cycle-tree. If a signed subgraph T' of T is also an Eulerian cycle-tree, then $T' = T$.*

Proof. The block cycles of T' are certainly block cycles of T. Suppose T' is properly contained in T. Then there exist a block cycle C of T' and a vertex u of C such that u is *not* a cut-vertex in T' but a cut-vertex in T. Let ε_T be a direction of T and $\varepsilon_{T'}$ a direction of T' such that they agree on C. Then Proposition 10(c) implies that $\varepsilon_T|T' = \varepsilon_{T'}$. Note that (C, ε_T)

must be coherent at u when C is considered as a block cycle in T', and must have either a sink or a source at u when C is considered as a block cycle in T. This is a contradiction. $\square$

Lemma 12 (Resolution of Indecomposable Flows). *Let f be a conformally indecomposable flow on (Σ, ε). Then f can be lifted to a conformally indecomposable flow $\tilde{f}$ on $(\tilde{\Sigma}, \tilde{\varepsilon})$. So $\Sigma(\tilde{f})$ is a cycle.*

Proof. It is clear that the conformal indecomposability of f implies that $\Sigma(f)$ is connected. Let $f = f_{(W,\varepsilon_f)}$, where (W, ε_f) is a directed closed positive walk of Σ. Lift (W, ε_f) to a directed closed walk $(\tilde{W}, \tilde{\varepsilon}_f)$ in $\tilde{\Sigma}$. Then f is lifted to a flow $f_{(\tilde{W},\tilde{\varepsilon}_f)}$ of $(\tilde{\Sigma}, \tilde{\varepsilon})$ by Proposition 7(a), denoted $\tilde{f} = f_{(\tilde{W},\tilde{\varepsilon}_f)}$. Suppose $\tilde{f}$ is decomposed into $\tilde{f} = \tilde{f}_1 + \tilde{f}_2$, where $\tilde{f}_i$ are nontrivial flows and $\tilde{f}_1 \tilde{f}_2 \geq 0$. Notice that

$$f = [\varepsilon, \varepsilon_f]\,|f|, \quad \tilde{f} = [\tilde{\varepsilon}, \tilde{\varepsilon}_f]\,|\tilde{f}|, \quad \tilde{f}_i = [\tilde{\varepsilon}, \tilde{\varepsilon}_f]\,|\tilde{f}_i|.$$

Denote $f_i = \pi(\tilde{f}_i)$, which are nontrivial flows of (Σ, ε). Then for each edge x of Σ,

$$\begin{aligned} f_i(x) &= [\tilde{\varepsilon}, \tilde{\varepsilon}_f](x^+)\,|\tilde{f}_i|(x^+) + [\tilde{\varepsilon}, \tilde{\varepsilon}_f](x^-)\,|\tilde{f}_i|(x^-) \\ &= [\varepsilon, \varepsilon_f](x)\big(|\tilde{f}_i(x^+)| + |\tilde{f}_i(x^-)|\big) \\ &= [\varepsilon, \varepsilon_f](x)\,\pi(|\tilde{f}_i|)(x). \end{aligned}$$

Taking absolute values of both sides, we obtain $|f_i| = \pi(|\tilde{f}_i|)$; subsequently, $f_i = [\varepsilon, \varepsilon_f]\,|f_i|$. Thus $f = f_1 + f_2$ and $f_1 f_2 = |f_1|\,|f_2| \geq 0$, meaning that f is conformally decomposable. This is a contradiction. $\square$

Theorem 13 (Classification of Indecomposable Flows). *Let f be a flow of (Σ, ε) and $\Omega = \Sigma(f)$. Then f is conformally indecomposable if and only if Ω is an Eulerian cycle-tree with a direction $\varepsilon_\Omega = \varepsilon_f|_\Omega$ and*

$$f = [\varepsilon, \varepsilon_\Omega]\, I_\Omega. \tag{22}$$

Proof. We show necessity first. Recall that ε_f is an orientation obtained from ε by reversing the orientations on the edges x such that $f(x) < 0$. Then $|f| = [\varepsilon, \varepsilon_f]\, f$ is a conformally indecomposable flow on (Σ, ε_f); so is a minimal flow because of nonnegativity. Let ε_f be lifted to an orientation $\tilde{\varepsilon}_f$ on $\tilde{\Sigma}$ and set $\tilde{\varepsilon}_\Omega = \tilde{\varepsilon}_f|\tilde{\Omega}$. Let $W = u_0x_1u_1x_2\ldots x_nu_n$ be a closed positive walk such that (W, ε_Ω) is directed. Then $|f| = f_W$ within (Σ, ε_f) by Corollary 5 and Lemma 8. Let W be lifted to a walk $\tilde{W} = u_0^{\alpha_0}\tilde{x}_1u_1^{\alpha_1}\tilde{x}_2\ldots\tilde{x}_nu_n^{\alpha_n}$ in $\tilde{\Sigma}$. Then $(\tilde{W}, \tilde{\varepsilon}_\Omega)$ is a directed closed walk in $\tilde{\Sigma}$ by Lemma 2, and $f_{\tilde{W}}$ is a conformally indecomposable flow of $(\tilde{\Sigma}, \tilde{\varepsilon}_f)$ by Lemma 12. Since $\tilde{\Sigma}$ is an unsigned graph, the closed walk $\tilde{W}$ must be a cycle. Since $\tilde{\Sigma}$ is a double covering of Σ and $\pi(\tilde{W}) = W$, it follows that W has only possible double vertices and double edges. If W has no double vertices, that is, W has no self-intersections, then (W, ε_Ω) is a directed cycle. Clearly, its underlying signed graph is a balanced cycle, of course is an Eulerian cycle-tree contained in Σ, and $f = [\varepsilon, \varepsilon_\Omega]\, I_\Omega$.

Assume that W has some self-intersections. Let u be a double vertex of W. Rewrite $W = W_1W_2$, where W_i are closed walks with the initial and terminal vertices at u; more specifically,

$$W_1 = u_0x_1u_1\ldots x_mu_m, \quad W_2 = u_mx_{m+1}u_{m+1}\ldots x_nu_n$$

with $u_0 = u_m = u_n = u$. We claim that W_i are negative, $(W_i, \varepsilon_\Omega)$ are incoherent at u and coherent elsewhere, and u is a cut-vertex.

Notice that $(W_i, \varepsilon_\Omega)$ are coherent everywhere except at u. Suppose $(W_1, \varepsilon_\Omega)$ is coherent at u. Then $(W_2, \varepsilon_\Omega)$ is also coherent at u. Thus $(W_i, \varepsilon_\Omega)$ are directed closed walks. We then have $f_W = f_{W_1} + f_{W_2}$ within (Σ, ε_f), meaning that $|f|$ is conformally decomposable; this is a contradiction. Hence $(W_i, \varepsilon_\Omega)$ must be incoherent at u. Lemma 3 implies that the closed walks W_i are negative. Let's write $\tilde{W} = \tilde{W}_1 \tilde{W}_2$, where

$$\tilde{W}_1 = u_0^{\alpha_0} \tilde{x}_1 u_1^{\alpha_1} \ldots \tilde{x}_m u_m^{\alpha_m}, \quad \tilde{W}_2 = u_m^{\alpha_m} \tilde{x}_{m+1} u_{m+1}^{\alpha_{m+1}} \ldots \tilde{x}_n u_n^{\alpha_n}$$

are open simple paths. Then $\alpha_0 = \alpha_n = -\alpha_m$ by Lemma 2.

Suppose u is not a cut-vertex, that is, the walks W_1 and W_2 meet at a vertex v other than u. Write the vertex v in W_1, W_2 as u_k, u_h respectively, that is, $v = u_k = u_h$, where $1 \leq k \leq m-1$ and $m+1 \leq h \leq n-1$. Since W is a cycle, then $u_k^{\alpha_k} \neq u_h^{\alpha_h}$; subsequently, $\alpha_k = -\alpha_h$. Consider the closed walk

$$\tilde{W}' = u_0^{\alpha_0} \tilde{x}_1 u_1^{\alpha_1} \ldots u_{k-1}^{\alpha_{k-1}} \tilde{x}_k u_k^{\alpha_k} (u_h^{-\alpha_h}) \tilde{x}_h^* u_{h-1}^{-\alpha_{h-1}} \ldots u_{m+1}^{-\alpha_{m+1}} \tilde{x}_{m+1}^* u_m^{-\alpha_m} (u_0^{\alpha_0}).$$

Let $s = \tilde{\varepsilon}_\Omega(u_0^{\alpha_0}, \tilde{x}_1)$. Then $\tilde{\varepsilon}_\Omega(u_0^{\alpha_k}, \tilde{x}_k) = -s$, for the open walk $(\tilde{W}_1, \tilde{\varepsilon}_\Omega)$ is directed. Analogously, $s = \tilde{\varepsilon}_\Omega(u_m^{\alpha_m}, \tilde{x}_{m+1}) = \alpha_m \varepsilon_\Omega(u_m, x_{m+1})$, the second equality is by definition (2) of lifting orientation. Then $\tilde{\varepsilon}_\Omega(u_m^{-\alpha_m}, \tilde{x}_{m+1}^*) = -\alpha_m \varepsilon_\Omega(u_m, x_{m+1}) = -s$, that is, $(\tilde{W}', \tilde{\varepsilon}_\Omega)$ is coherent at $u_0^{\alpha_0}$. Thus $(\tilde{W}', \tilde{\varepsilon}_\Omega)$ is a directed closed walk in $\tilde{\Omega}$. It follows that (W', ε_Ω) is a directed closed walk in Ω, and is contained in the walk W as edge multisets. Therefore, $f_{W'} \leq f_W$. The minimality of f_W implies that $f_{W'} = f_W$; this is impossible, for W' is properly contained in W as multisets.

Now the closed walk W has only possible double vertices and double edges, and all double vertices are cut-vertices. The signed subgraph Ω is obtained from the cycle $\tilde{W}$ by the projection, identifying some pairs of vertices and some pairs of edges. The unidentified vertices and edges form the block cycles C_i, and the connected components of identified vertices and edges form the block paths P_j (of possible zero length) between some pairs of the cycles C_i. More specifically, viewing each connected component of identified edges and vertices as a single *path-identification*, then each path-identification transforms a cycle exactly into two cycles, the identified path (of possible zero length) connects the two cycles and becomes a block path. Since each double vertex of W is a cut-vertex of Ω, it turns out that Ω is a tree-like with "vertices" C_i and "edges" P_j; the cycles C_i must be blocks, the vertices and edges of P_j are cut-vertices and cut-edges respectively.

Recall the incoherence of $(W_i, \varepsilon_\Omega)$ at the double vertex u. It follows that $(C_i, \varepsilon_\Omega)$ are incoherent at the cut-vertices of Ω on C_i and coherent elsewhere. Thus the cycles C_i has sign $(-1)^p$ by Lemma 2, where p is the number of cut-vertices on C_i. We have finished proof that Ω is an Eulerian cycle-tree, ε_Ω is a direction of Ω, and (W, ε_Ω) is a directed Eulerian tour. Since each edge on the block cycles appears once in W and each edge on the block paths appears twice in W, we see that $f_W = I_\Omega$ within (Σ, ε_f). Therefore $f = f_{(W, \varepsilon_\Omega)} = [\varepsilon, \varepsilon_\Omega]\, I_\Omega$.

Conversely for sufficiency, let's write $[\varepsilon, \varepsilon_\Omega]\, I_\Omega = \sum_{i=1}^k f_i$, where f_i are conformally indecomposable flows of (Σ, ε). Let $(\Omega_i, \varepsilon_{\Omega_i})$ be Eulerian cycle-trees such that $f_i = [\varepsilon, \varepsilon_{\Omega_i}]\, I_{\Omega_i}$. Since each f_i conforms to the sign pattern of f, then Ω_i is a signed subgraph of Ω and ε_{Ω_i} is the restriction of ε_Ω on Ω_i. The block cycles of Ω_i are certainly block cycles of Ω. Consider a possible block path P of Ω_i. If $\ell(P) = 0$, then P is the intersection of two block cycles C_1, C_2 of Ω_i. Clearly, the block cycles C_1, C_2 of Ω_i must be block cycles of Ω, and P is certainly contained in Ω. If $\ell(P) > 0$, then $f_i(x) = \pm 2$ for all edges x of P. Since f_i conforms to the sign pattern of f, it forces that $[\varepsilon, \varepsilon_\Omega]\, I_\Omega(x) = \pm 2$ for all edges x of P. This means that P

is a block path of Ω. This means that Ω_i is an Eulerian cycle-tree of Ω. Thus $\Omega = \Omega_i$ by Lemma 11. Therefore $k = 1$ and $f_1 = [\varepsilon, \varepsilon_\Omega]\, I_\Omega$ is conformally indecomposable. $\square$

Proposition 14. *A signed graph Ω is prime Eulerian if and only if Ω is an Eulerian cycle-tree.*

Proof. If Ω is an Eulerian cycle-tree, then by Proposition 10 there exist an orientation ε_Ω on Ω and a closed walk that uses every edge of Ω once but at most twice such that (W, ε_Ω) is a directed closed positive walk. It is clear from the structure of Eulerian cycle-tree that (W, ε_Ω) does not contain properly directed closed positive subwalks. So Ω is a prime Eulerian signed graph.

Conversely, if Ω is a prime Eulerian signed graph, then by definition there exist an orientation ε_Ω on Ω and a closed walk W that uses every edge of Ω at least once but at most twice, such that (W, ε_Ω) is a directed closed positive walk and does not properly contain directed closed positive subwalks. Since f_W is a flow of $(\Omega, \varepsilon_\Omega)$, there is a conformally indecomposable flow f such that $f_W \geq f \geq 0$. By Theorem 13 there exist an Eulerian cycle-tree T and its direction ε_T such that $f = [\varepsilon_\Omega, \varepsilon_T] I_T$. Then we must have $\varepsilon_T = \varepsilon_\Omega$ on T. Let (W_1, ε_T) be a directed closed positive walk on T that follows the direction of W. Then (W_1, ε_T) is a directed closed positive subwalk of (W, ε_Ω). The primeness of (W, ε_Ω) implies that $(W, \varepsilon_\Omega) = (W_1, \varepsilon_T)$. Hence $\Omega = T$. $\square$

Theorem 15 (Half-integer Scale Decomposition)**.** *Let Ω be an Eulerian cycle-tree with a direction ε_Ω. If Ω is not a signed-graph circuit, then there exists a closed positive walk*

$$W = C_0 P_1 C_1 P_2 \dots P_n C_n P_{n+1}, \quad n \geq 1$$

on Ω, satisfying the four conditions.

(a) *C_i are all end-block cycles, and P_i are paths of positive lengths.*
(b) *Each edge of non-end block cycles appears in exactly one of P_i, and each edge of block paths appears in exactly two of P_i.*
(c) *Each $(C_i P_{i+1} C_{i+1}, \varepsilon_\Omega)$ is a directed circuit of Type III with $C_{n+1} = C_0$.*
(d) *$I_\Omega = \frac{1}{2} \sum_{i=0}^{n} I_{\Sigma(C_i P_{i+1} C_{i+1})}$.*

Proof. Let $C_0, C_1, \dots, C_n$ be end-block cycles with unique cut-vertices $u_0, u_1, \dots, u_n$ respectively. Let Q_j be block paths, and R_k the paths on non-end block cycles between two next cut-vertices. Let C_i be lifted to two disjoint directed open paths $\tilde{C}_i^+, \tilde{C}_i^-$ between u_i^+ and u_i^-. Let Q_j be lifted to two disjoint directed open paths $\tilde{Q}_j^+, \tilde{Q}_j^-$. And let R_k be lifted to disjoint directed open paths $\tilde{R}_k^+, \tilde{R}_k^-$. Each $\tilde{C}_i^\alpha$ connects exactly two paths $\tilde{Q}_j^+, \tilde{Q}_j^-$ and $\tilde{Q}_j^+ \tilde{C}_i^\alpha \tilde{Q}_j^-$ is a directed open path. Likewise, each $\tilde{R}_k^\gamma$ connects exactly two paths $\tilde{Q}_j^\beta, \tilde{Q}_{j'}^{\beta'}$ with $j \neq j'$ and $\tilde{Q}_j^\beta \tilde{R}_k^\gamma \tilde{Q}_{j'}^{\beta'}$ is a directed open path; subsequently, $\tilde{Q}_j^{-\beta} \tilde{R}_k^{-\gamma} \tilde{Q}_{j'}^{-\beta'}$ is a directed open path.

Let us partition the collection $\{\tilde{R}_k^+, \tilde{R}_k^-\}$ of open paths into two disjoint sub-collections $\{\tilde{R}_k^{\gamma_k}\}$ and $\{\tilde{R}_k^{-\gamma_k}\}$ arbitrarily. Then the union of the paths in $\{\tilde{Q}_j^+, \tilde{Q}_j^-, \tilde{R}_k^{\gamma_k}\}$ is a collection of disjoint directed open paths, having initial and terminal vertices $u_i^\alpha, u_{i'}^{\alpha'}$ with $i \neq i'$. Let $\tilde{P}_1$ denote that the directed open path with from a vertex $u_0^{\beta_0}$ and its terminal vertex $u_{s_1}^{\gamma_1}$; let $\tilde{P}_2$ denote the open path from $u_{s_1}^{\beta_1}$ ($\beta_1 = -\gamma_1$) to $u_{s_2}^{\gamma_2}$; and let $\tilde{P}_3$ denote the open path from $u_{s_2}^{\beta_2}$ ($\beta_2 = -\gamma_2$) to $u_{s_3}^{\gamma_3}$. Continue this procedure, we obtain a sequence of directed open paths

$$\tilde{P}_1,\ \tilde{P}_2,\ \dots,\ \tilde{P}_{m+1},$$

where $\tilde{P}_i$ is from $u_{s_{i-1}}^{\beta_{i-1}}$ to $u_{s_i}^{\gamma_i}$ with $\beta_i = -\gamma_i$, and $u_{s_{m+1}}^{\gamma_{m+1}} = u_0^{\gamma_0}$ with $\beta_0 = -\gamma_0$. Note that both $\tilde{P}_i \tilde{C}_{s_i}^{+} \tilde{P}_{i+1}$ and $\tilde{P}_i \tilde{C}_{s_i}^{-} \tilde{P}_{i+1}$ are directed open paths for all i from 0 to n.

Without loss of generality, let us denote by $\tilde{P}_1$ the open path from $u_0^{s_0}$ to $u_1^{t_1}$,

Let us partition the collection $\{\tilde{C}_i^{+}, \tilde{C}_i^{-}\}$ of open paths into two disjoint sub-collections $\{\tilde{C}_i^{\alpha_i}\}$ and $\{\tilde{C}_i^{-\alpha_i}\}$ arbitrarily. It follows that

$$\tilde{W} = \tilde{C}_0^{\alpha_0} \tilde{P}_1 \tilde{C}_1^{\alpha_1} \tilde{P}_2 \ldots \tilde{P}_n \tilde{C}_n^{\alpha_n} \tilde{P}_{n+1}$$

is a directed closed walk in $(\tilde{\Sigma}, \tilde{\varepsilon})$. Note that the projections of $\tilde{C}_i^{\alpha_i}$ are unbalanced cycles C_i in Σ, and (C_i, ε) is coherent everywhere except incoherence at u_i. Let P_i be the projection of $\tilde{P}_i$. Then the projection of $\tilde{W}$ is the directed closed walk

$$W = C_0 P_1 C_1 P_2 \ldots C_n P_{n+1}$$

in (Σ, ε), and each $(C_i P_{i+1} C_{i+1}, \varepsilon)$ is a directed circuit of Type III with $C_{n+1} = C_0$. Clearly, each edge of non-end-block cycles appears in exactly one of the paths P_i, and each edge of block paths appears in exactly two of the paths P_i. Hence $I_\Omega = f_W$ and

$$I_\Omega = I_{C_0} + I_{P_{n+1}} + \sum_{i=1}^{n} \big(I_{C_i} + I_{P_i} \big) = \frac{1}{2} \sum_{i=0}^{n} I_{\Sigma(C_i P_{i+1} C_{i+1})}.$$

$\square$

The weights on the edges of the cycle-tree in Figure 4 form a conformally indecomposable flow with respect to the given direction there, which is exactly the characteristic vector of the cycle-tree. This conformally indecomposable flow can be decomposed into one-half of the sum of three signed-graph circuit flows as demonstrated in Figure 5.

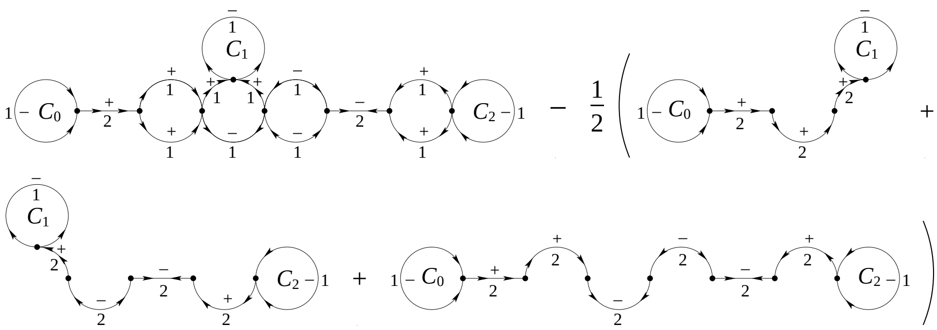


FIGURE 5. A conformally indecomposable flow is decomposed conformally into *one-half* of signed-graph circuit flows of Type III.

The half-integer phenomenon in Theorem 15(d) is also appeared in a result of Geelen and Guenin [9] (Corollary 1.4, p. 283), though our problem of classifying conformally indecomposable flows and certain optimal problem considered by Geelen and Guenin for signed graphs are completely different. The half-integer phenomenon in both cases is a consequence of sign on the edges of signed graphs.

One may consider decomposition of integral flows without conforming the sign patterns. It is clear that every nontrivial integral flow is a positive integral linear combination of conformally indecomposable flows. We shall see that each conformally indecomposable flow can be further decomposed into integral linear combination of signed-graph circuit flows

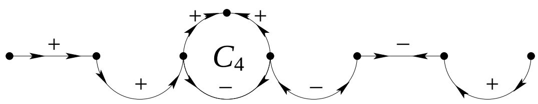


FIGURE 6. A maximal independent set of the Eulerian cycle-tree in Fig. 4

without confirming the sign patterns. So each integral flow is an integral linear combination of signed-graph circuit flows. This fact is already explicitly given in terms of a maximal independent set (=matroid basis) in [5] (see Eq. (4.7) of Theorem 4.9, p. 275). For instance, the signed graph in Figure 6 is an maximal independent set of the Eulerian cycle-tree in Figure 4. The conformally indecomposable flow in Figure 4 is further decomposed into signed-graph circuit flows in Figure 7 without confirming the sign patterns. We summarize the fact into the following Corollary 16.

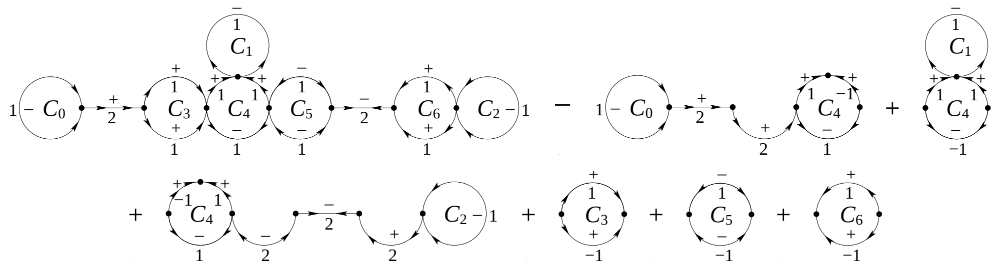


FIGURE 7. A conformally indecomposable flow is decomposed non-conformally into signed-graph circuit flows.

Corollary 16. *(a) If f is a nontrivial integral flow of an oriented singed graph, then $2f$ can be conformally decomposed into a positive integral linear combination of signed-graph circuit flows.*

(b) Every nontrivial integral flow of an oriented signed graph can be decomposed (perhaps conformally perhaps non-conformally) into a positive integral linear combination of some signed-graph circuit flows.

The following proposition is trivial but explains why there are exactly three natural types of signed-graph circuits introduced by Zaslavsky [12, 14].

Proposition 17. *Let Ω be a signed graph. Then the following statements are equivalent.*

(a) Ω is a minimal Eulerian signed graph.
(b) Ω is a minimal Prime Eulerian signed graph.
(c) Ω is a minimal Eulerian cycle-tree.
(d) Ω is a signed-graph circuit.

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